Hybridisable discontinuous Galerkin methods

Professor Bernardo Cockburn, Dr Ngoc-Cuong Nguyen and Professor Jaime Peraire have collaborated to find more efficient means of simulating complex phenomena. Here, they discuss their research into hybridisable discontinuous Galerkin methods and their potential applications.

Can you begin by outlining where your individual initial interests in solving challenging partial differential equations stemmed from?

BC: I became interested in the secondary recovery methods of extracting petroleum from underground in order to determine total oil production.

NCN: I wanted to understand the propagation of electromagnetic waves to resolve the radar cross-section of how detectable an object is with radar.

JP: Initially, I developed a desire to be able to model and simulate physical phenomena of interest such as the flow over wings to understand the forces that the air exerts on aircraft.

What are Galerkin methods and what are the differences between the continuous and discontinuous forms?

Galerkin methods provide a weak formulation to discretise the physical laws governing the problem of interest in such a way that they can be represented on a computer. In Galerkin methods, the approximation spaces for the test functions and approximate solutions are the same. When the approximation spaces are continuous, the methods are called continuous Galerkin (CG) methods, and discontinuous Galerkin (DG) methods when the approximation spaces are discontinuous.

In which areas are Galerkin methods applied and how is your research adding to the body of knowledge on them?

The use of Galerkin methods spans across a wide range of applications, but perhaps most notably in computational fluid dynamics, solid mechanics and electromagnetics. We are currently developing a specific form of Galerkin methods that are extremely efficient and accurate and will increase their range of applications.

What advantages do DG methods have over more traditional finite difference and finite volume methods?

DG methods are mathematically more rigorous and, as a result, are easy to study and generalise. In contrast with finite volume methods, DG methods of arbitrarily high-order accuracy on very general meshes can be easily obtained. In comparison to finite difference methods, DG methods can be more localised and easily used with adaptive algorithms. Moreover, DG methods provide a systematic way to deal with a wide range of partial differential equations with a high level of accuracy.

Which hybridisable discontinuous Galerkin (HDG) methods have you shown to actually be subclasses of the old DG method? What is the importance of hybridisation and were hybrid methods developed in complete separation from HDG methods?

Certain classes of well-known DG methods can be viewed as (or are equivalent to) HDG methods. However, there are HDG methods that constitute new DG methods. The hybridisation of the so-called mixed methods was introduced in the mid-1960s in order to be able to efficiently implement them. Applying this technique to an old DG method gives rise to an HDG method.

The HDG methods were devised so that their hybridisation was ensured by construction. Hybrid methods were also introduced in the mid-1960s, but did not have all the stabilisation and accuracy properties of the HDG methods.

For which practical problems have you defined and numerically tested your HDG methods? How do you test their accuracy?

We have defined and tested our methods for a wide range of practical problems such as convection-diffusion systems, elasticity, Maxwell’s equations and the eikonal equation. We usually test the accuracy of our methods through computing approximations, particularly for problems for which an exact solution is known.

Could you explain the advantages of collaborating with computational mathematicians, aerospace engineers and civil engineers from around the US?

Collaboration with computational mathematicians brings about useful ideas from mathematics to tackle real-world applications in engineering and science. As a result, the ensuing new mathematical methods can be used by engineers to solve engineering problems more effectively and rigorously than traditional approaches. Likewise, working with engineers helps mathematicians understand and maintain their focus on relevant physical problems. The outcome of the collaboration is then guaranteed to be applicable to real-life issues.

Where would you like to focus your research efforts in the coming years?

In terms of applications, we would like to develop simulation methods for embedded autonomous systems and real-time applications. We are also interested in creating large eddy simulations and aeroacoustics using HDG methods. From a theoretical point of view, we would like to develop a systematic way of ensuring superconvergence, or optimal convergence, for general meshes for a wide variety of partial differential equations.
Capturing complex phenomena

Researchers from the Massachusetts Institute of Technology and University of Minnesota have collaborated to develop more effective and efficient methods of programming computer simulations – findings which are of benefit to many disciplines.

If, for instance, a structural engineer designs a bridge and wants to understand its possible loading, he might wish to investigate how the weight of vehicles travelling across it will be distributed. However, this is not a straightforward process for an unusual shape such as a bridge because, although the equations involved can be derived, there are no direct means of solving them.

It is therefore often necessary to use the finite element method, a numerical technique that enables approximate solutions. In a complex case like the bridge example, it may be easier to solve by breaking the problem down into small polyhedra called elements, which constitute what is known as the mesh. Next, the form for the approximations inside the elements needs to be decided (constant, linear or quadratic functions, etc.), and whether these approximations are continuous or discontinuous. Finally, a means of capturing the physical law under consideration is required. Once these stages are achieved, the weight distribution can be calculated for each section and then related to the others to obtain a good approximation of the loadings the real bridge would experience.

Complex Computations
Although the finite element method is extremely useful for solving partial differential equations, the equations must first be transferred into their discrete counterparts – a process known as discretisation – in order to be solved. However, discretising continuous data results in errors that propagate and grow through the solution. It is therefore necessary to ensure that the numerical method used yields the smallest possible deviation between correct and incorrect results; denoted by the order of accuracy.

Unfortunately, a high order of accuracy requires more computations, rendering the modelling process time-consuming and potentially inefficient. Scientists, mathematicians and engineers have therefore been working hard to find numerical methods that are both accurate and computationally inexpensive for capturing any given physical law. Methods that can achieve this balance will have far-reaching benefits, with dramatic impact upon many fields.

Galerkin Methods
Since the physicist Walther Ritz discovered Galerkin methods over a century ago, the drive for computational efficiency has led to the application of discontinuous Galerkin (DG) methods. Their inherent stability ensures a high order of accuracy and makes them ideal for approximating quantities which could change in magnitude when varying from one element to another. However, despite their apparent suitability for use in computational modelling, they have been shown to be inefficient for a range of physical laws and so their potential has yet to be fully realised.

Now, researchers from the University of Minnesota and Massachusetts Institute of Technology have collaborated with the intention of finding highly efficient variations of Galerkin methods that can be more widely applied. Professor Bernardo Cockburn, Dr Ngoc-Cuong Nguyen and Professor Jaime Peraire have focused much of their work on the use of hybridisable discontinuous Galerkin (HDG) methods and the ways they can be employed without generating too many additional computations.

Their studies thus far have resulted in the publication of several papers demonstrating the efficacy of using HDG methods for a variety of applications including computational fluid dynamics, continuum mechanics and electromagnetism. Indeed, examples of physical phenomena they have applied their studies to thus far include convection-diffusion, acoustic and elastic wave propagation, nonlinear hyperbolic conservation laws, and large deformation elasticity and high-frequency wave propagation.

Advantages of HDG
Some notable advantages to using HDG methods have been found. Unlike other DG methods, HDG methods produce a reduced number of globally coupled degrees of freedom, which can lead to a reduction in both the number of computations required and the amount of computational memory used. Additionally, HDG methods have been found to provide superconvergence – that is, methods that converge faster than might be expected. In particular, they are more precise than standard DG methods.

The team has also shown that the HDG methods they have developed remain parallelisable and therefore conducive to efficient computation. Indeed, the problems they seek to solve are formulated at a level whereby they can be solved independently, increasing the amount of computations that can be done simultaneously and at the same

Scalar wave equation. High-order accurate approximation of the wave field scattered from a kite-shaped object for an incident wave travelling from left to right.
SUPERCONVERGENT HYBRIDISABLE DISCONTINUOUS GALERKIN METHODS FOR PARTIAL DIFFERENTIAL EQUATIONS

OBJECTIVE
To find new superconvergent hybridisable discontinuous Galerkin methods that are more efficient and able to capture more complex physical phenomena.

KEY COLLABORATORS
Professor Jay Gopalakrishnan, Portland State University, USA • Professor Raytcho Lazarov, Texas A&M University, USA • Professor Chi-Wang Shu, Brown University, USA

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CONTACT
Professor Bernardo Cockburn
Professor of Mathematics
School of Mathematics
University of Minnesota
127 Vincent Hall
206 Church Street SE
Minneapolis, MN 55455
USA
T +1 612 625 2587
E cockburn@math.umn.edu
www.math.umn.edu/~bcockburn

PROFESSOR BERNARDO COCKBURN received his PhD from the University of Chicago in 1986 under the direction of Jim Douglas, Jr. In 1987, he joined the School of Mathematics at the University of Minnesota and became a McKnight Distinguished University Professor in 2007.

DR NGOC-CUONG NGUYEN is a principal research scientist in the Department of Aeronautics and Astronautics at the Massachusetts Institute of Technology and also affiliated with the Center for Computational Engineering’s School of Engineering.

PROFESSOR JAIME PERAIRE is the Department Head and H.N. Slater Professor of Aeronautics and Astronautics at the Massachusetts Institute of Technology. Prior to this, he was a member of the faculty at the University of Wales and at Imperial College London, UK.

A BRIEF HISTORY OF GALERKIN METHODS

Although it was the Swiss mathematician Walther Ritz who first discovered the method, it was actually Soviet engineer Boris Galerkin who introduced the approach that became his eponymous technique in 1915, using continuous approximations in a distinctive way that enabled him to capture the particular physical law under consideration.

It was not until 1973 that William H Reed and Thomas R Hill demonstrated how to use discontinuous approximations to discretise the equations separately on each element.

By the end of the 20th Century, Professors Bernardo Cockburn and Chi-Wang Shu – through their work on nonlinear hyperbolic systems – had developed discontinuous Galerkin methods to the point where they were versatile, stable and high-order accurate, and therefore part of the mainstream computational method toolkit. However, their inefficiency when applied to steady-state problems arising in heat conduction, time-harmonic electromagnetism or solid mechanics rendered them unsuitable for many applications.

This undesirable feature was overcome in the late 2000s, when Cockburn, alongside Professors Jay Gopalakrishnan and Raytcho Lazarov, introduced hybridisable discontinuous Galerkin methods. This new set of techniques was shown to be more accurate than other discontinuous methods for steady-state problems and, in some cases, approximated the solutions better than expected.

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